Introduction to Machine Learning

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A little bit about me...

LEONARDO HELICOPTERS

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RICARDO

KONGSBERG

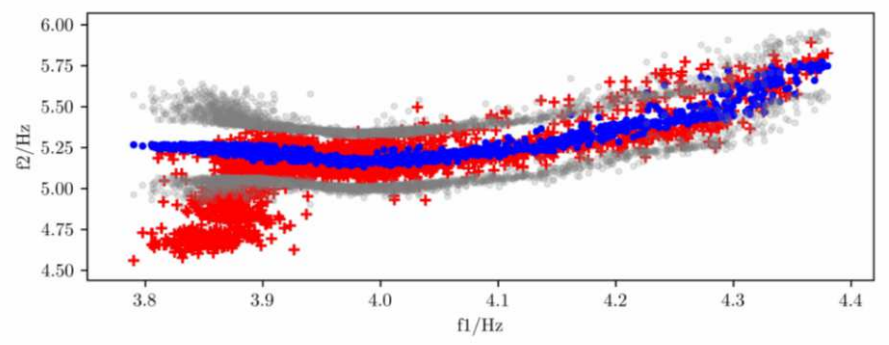
Callsign

Prenscia HBM

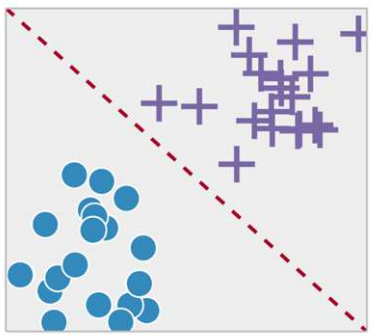
University of Strathclyde

The tools of machine learning

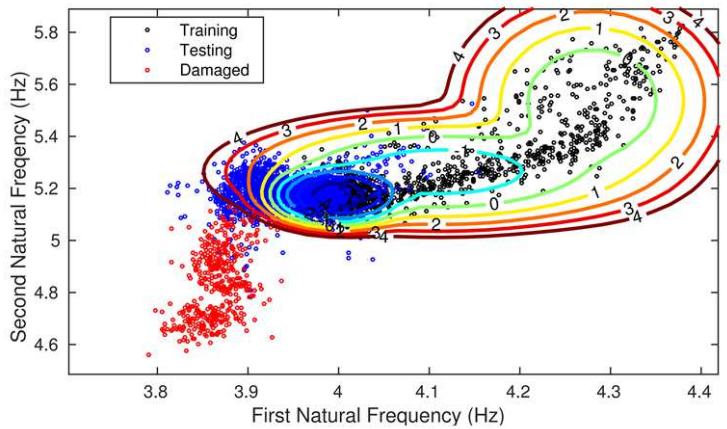
Regression



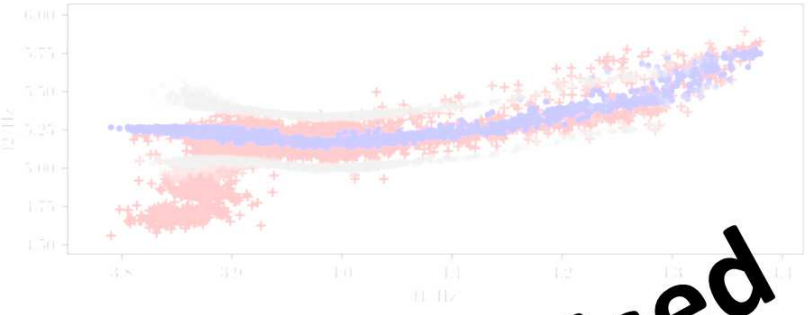
Classification



Density Estimation



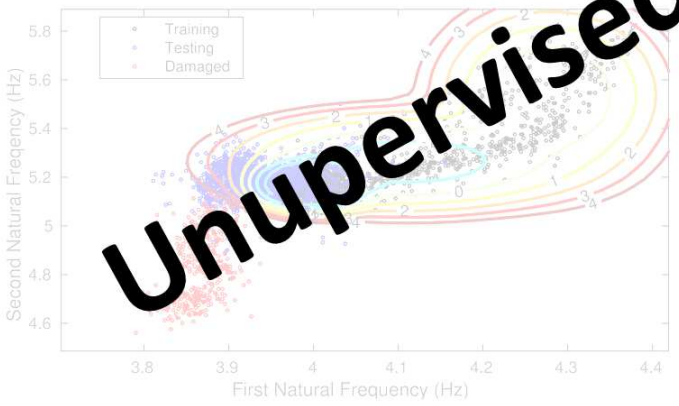
The tools of machine learning



- Classification

Supervised

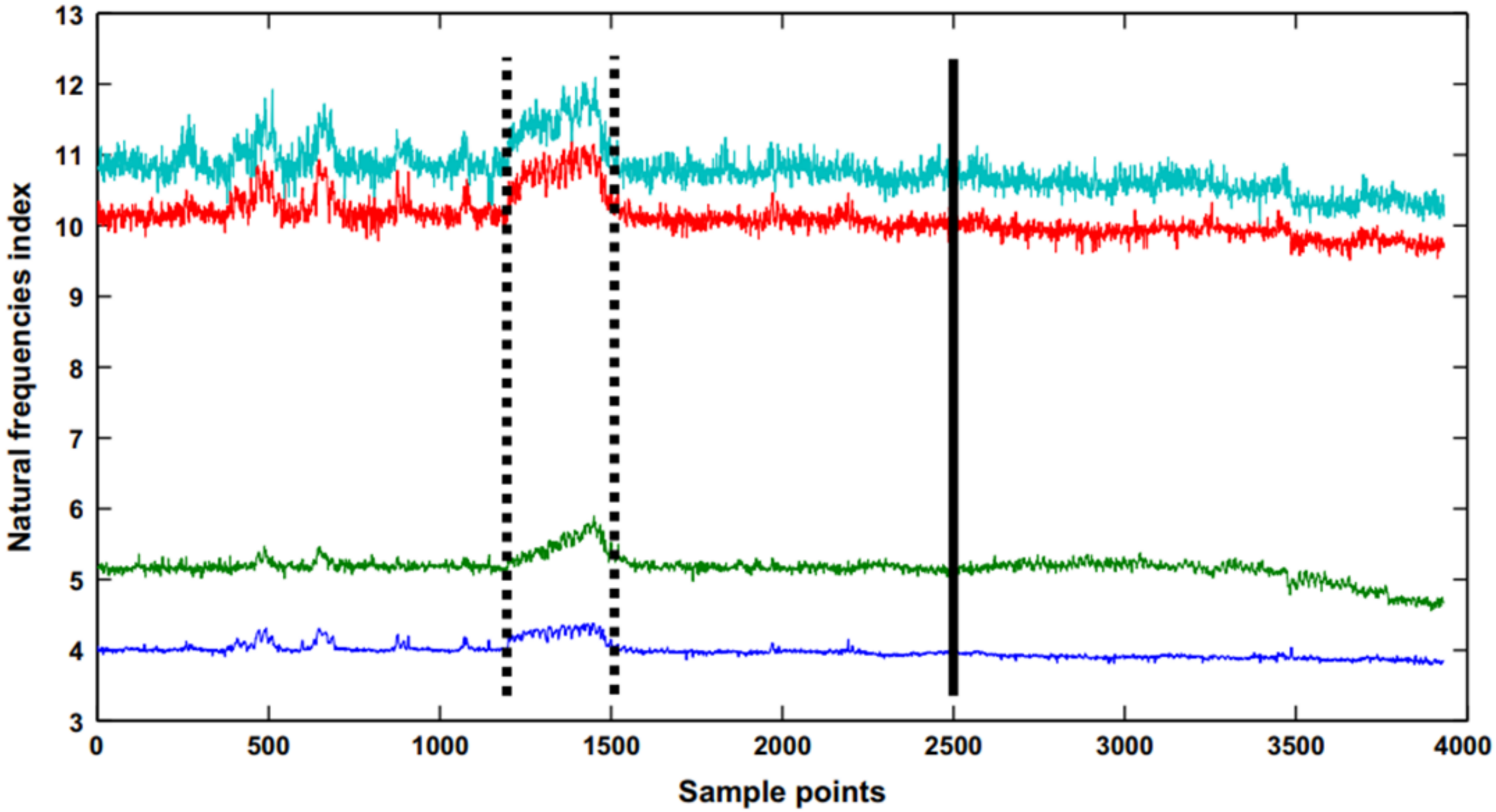
- Density Estimation



Unupervised

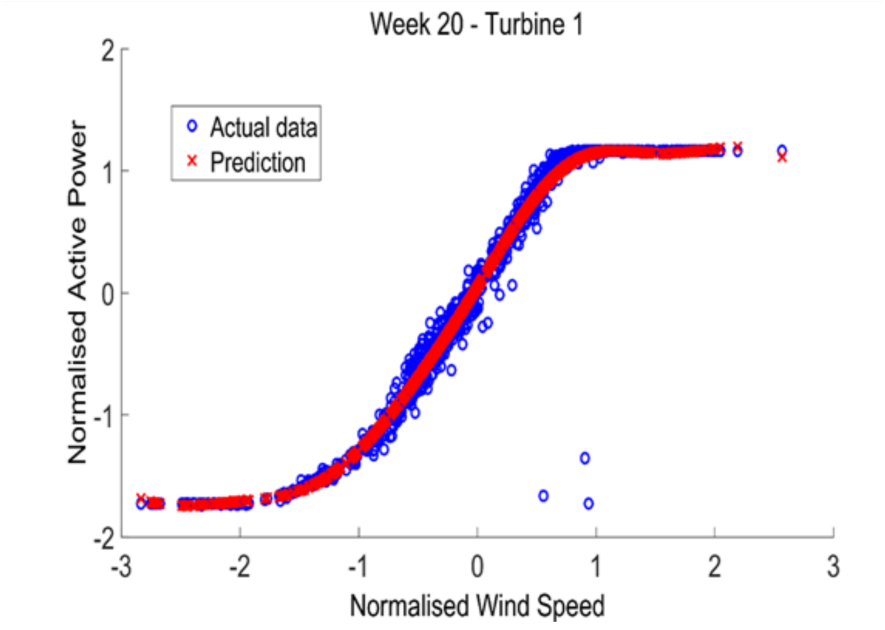
Motivating problem

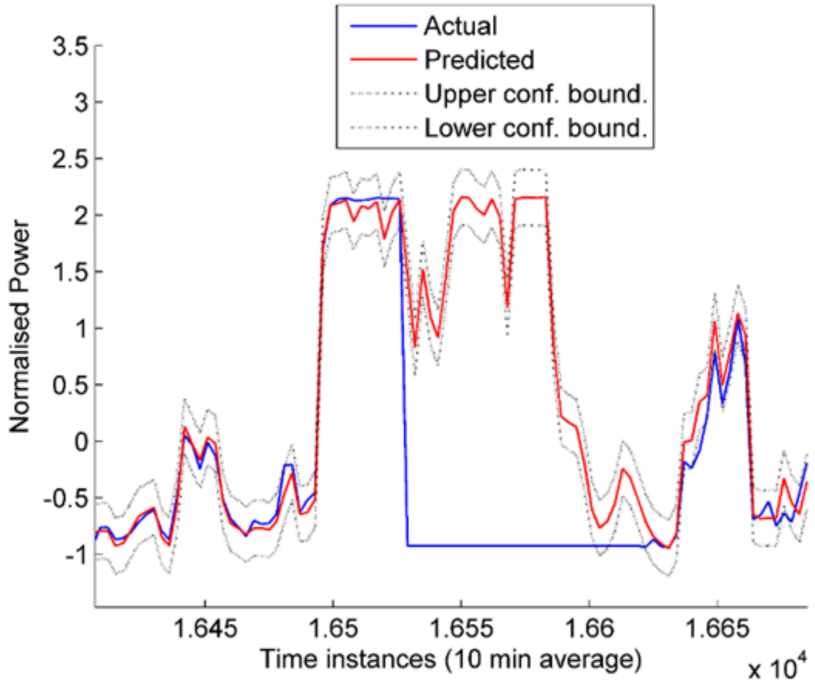
Given a set of measured natural frequencies from a bridge, can we detect damage?



Motivating problem

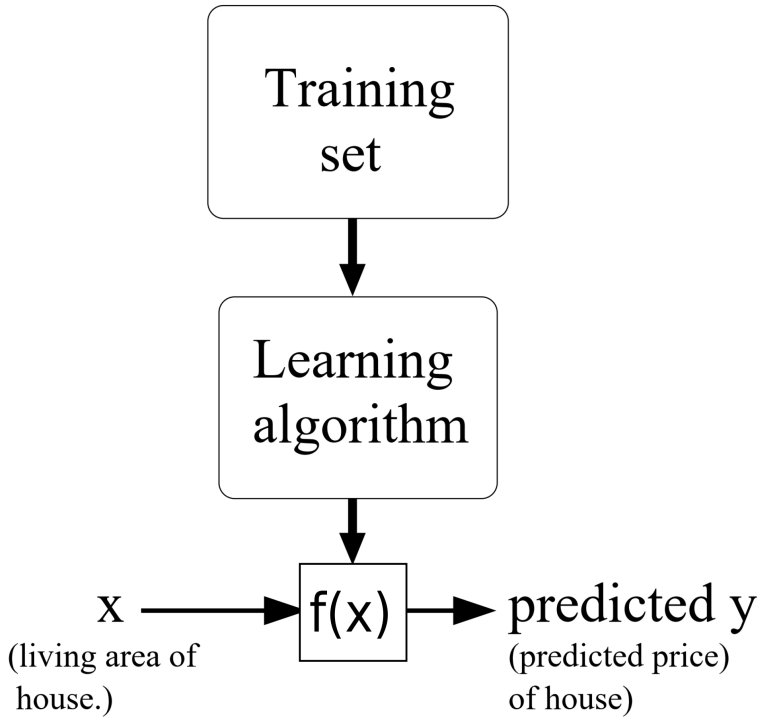
Can we diagnose problems on a wind turbine,
given measured wind and power?





Supervised learning

Supervised learning deals with the problem of modelling the relationship between a set of inputs, **x** and outputs **y**

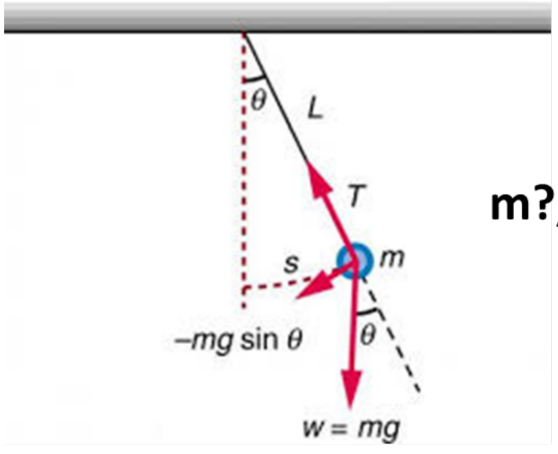


A first look at supervised learning: linear regression

Given some measured data,

there are generally two problems of interest:

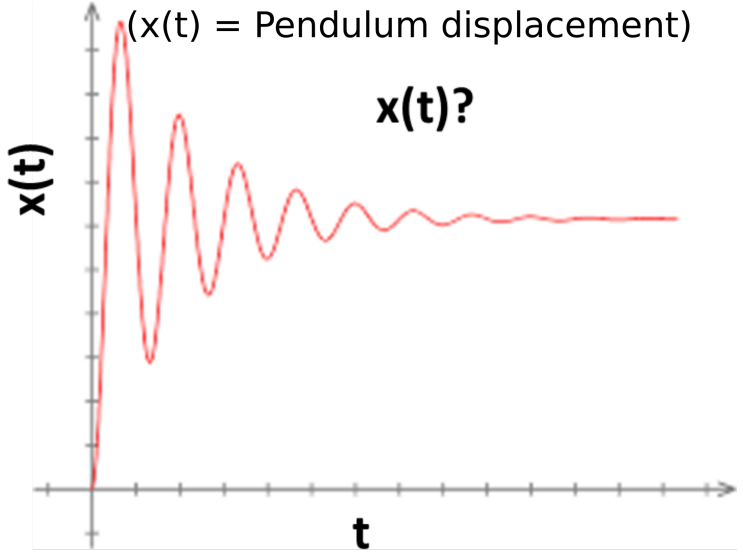
Identifying a system



m?, L?

Making predictions

(x(t) = Pendulum displacement)



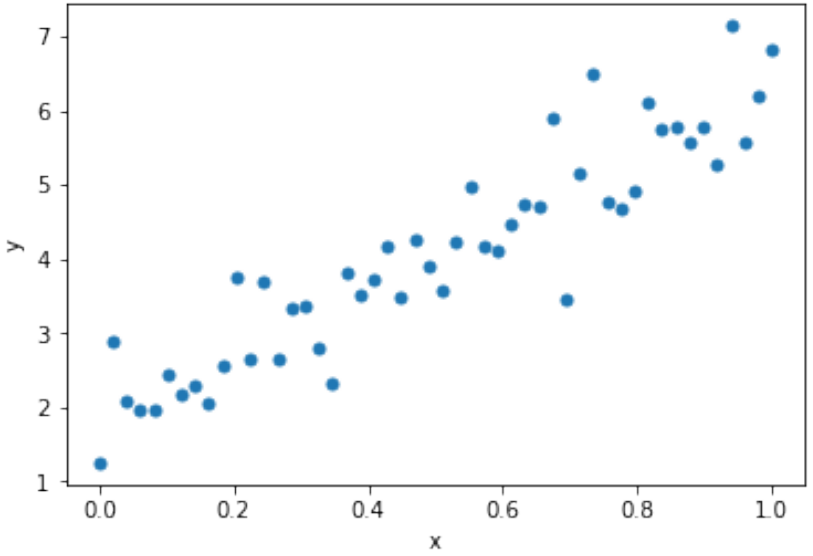
Linear Regression

Lets start with a simple toy example: a noisy $y = 5x + 1$

Can we learn the relationship between ***x*** and ***y*** from the data ?

We need to things:

1. A model
2. A loss function, that quantifies our error or predictive performance



Linear Regression

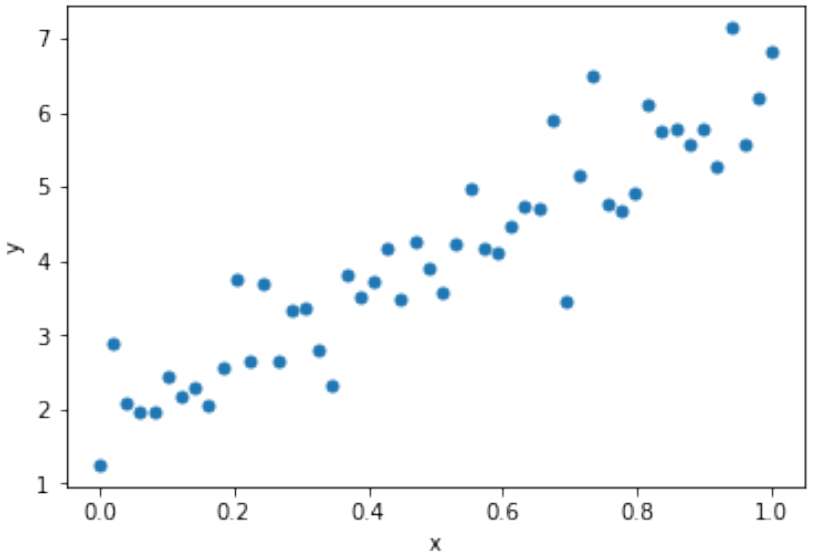
We need a model for the underlying function:

$$y = f(x)$$

Linear regression models $f(\mathbf{x})$ as:

$$y = x_0 w_0 + x_1 w_1 + x_2 w_2 + \ldots$$

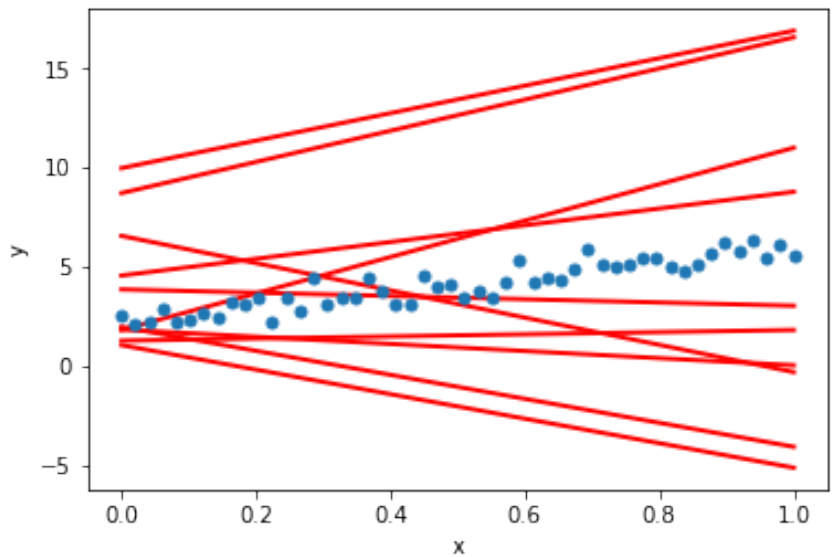
$$y = \sum_j x_j w_j$$



Linear Regression

So, we have a model, parametrised by $\mathbf{w}$, we now need to define a loss function so we can pick our weights appropriately

A bunch of models drawn at random, which fits the data best?

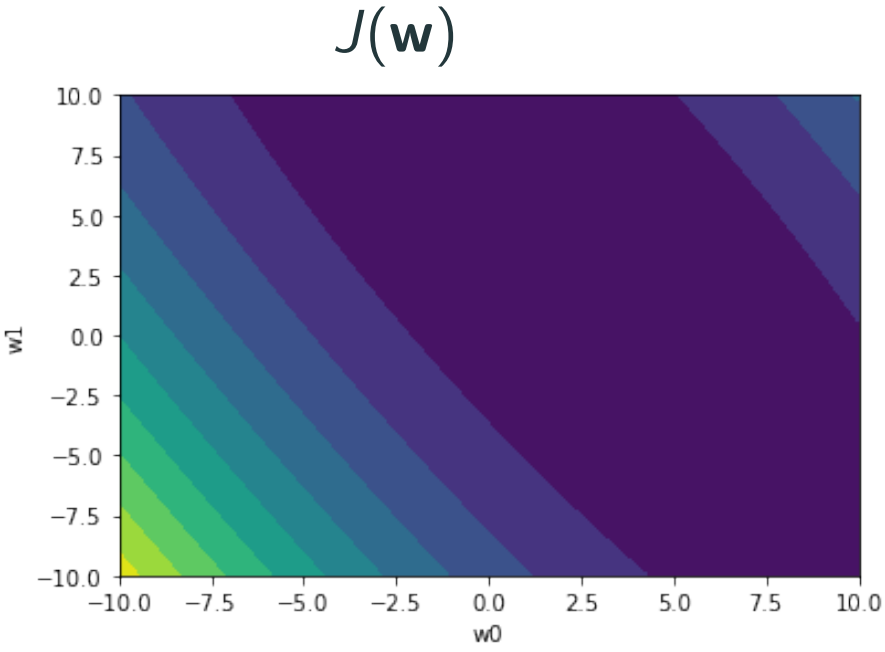


Squared error loss

One appropriate loss function is the mean of the squared prediction error:

$$J(\mathbf{w}) = \frac{1}{2}\sum_{i=1}^{n}(y^{(i)} - f(x^{(i)}))^2$$

But how should we find the value of $\mathbf{w}$ that minimises $J(\mathbf{w})$?



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- The idea is that we'll start with an initial choice for $\mathbf{w}$ and improve it iteratively in a direction that decreases $J(\mathbf{w})$ - our cost function
- We take a step in the direction of the gradient $\frac{\partial J}{\partial \mathbf{w}}$

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and remember,

$$f(\mathbf{x}^{(i)}) = \sum_{j}^{d} w_j x_j^{(i)}$$

Gradient Descent

Lets derive $\frac{\partial}{\partial w_j} J(\mathbf{w})$ for the case where we have a single training example $x_j^{(i)}$ $y^{(i)}$,

$$\begin{aligned}
\frac{\partial}{\partial w_j} J(\mathbf{w}) &= \frac{\partial}{\partial w_j} \frac{1}{2} (y^{(i)} - f(\mathbf{x}^{(i)}))^2 \\
&= 2 \frac{1}{2} (y^{(i)} - f(\mathbf{x}^{(i)})) \frac{\partial}{\partial w_j} (y^{(i)} - f(\mathbf{x}^{(i)})) \\
&= (y^{(i)} - f(\mathbf{x}^{(i)})) \frac{\partial}{\partial w_j} \Big(\sum_{k=0}^{d} x_k^{(i)} w_k - y^{(i)} \Big) \\
&= (y^{(i)} - f(\mathbf{x}^{(i)})) x_j^{(i)}
\end{aligned}$$

Gradient Descent, learning rule

We now have an update rule, that we can apply whenever we encounter a new observation,

$$w_j = w_j + \alpha\big(y^{(i)} - f(\mathbf{x}^{(i)})\big)x_j^{(i)}$$

here, α is a *learning rate*

Batch Gradient Descent

When we have all training observations $x^{(1)}, ..., x^{(n)}$ and $y^{(i)}, ..., y^{(n)}$, we can assemble this into an algorithm

while not converged **do**
  for every j, **do**
    $w_j \leftarrow w_j + \alpha \sum_{i=1}^{n} \left(y^{(i)} - \sum_j^d x_j^{(i)} w_j\right) x_j$
  end for
end while

this is called *batch gradient descent*

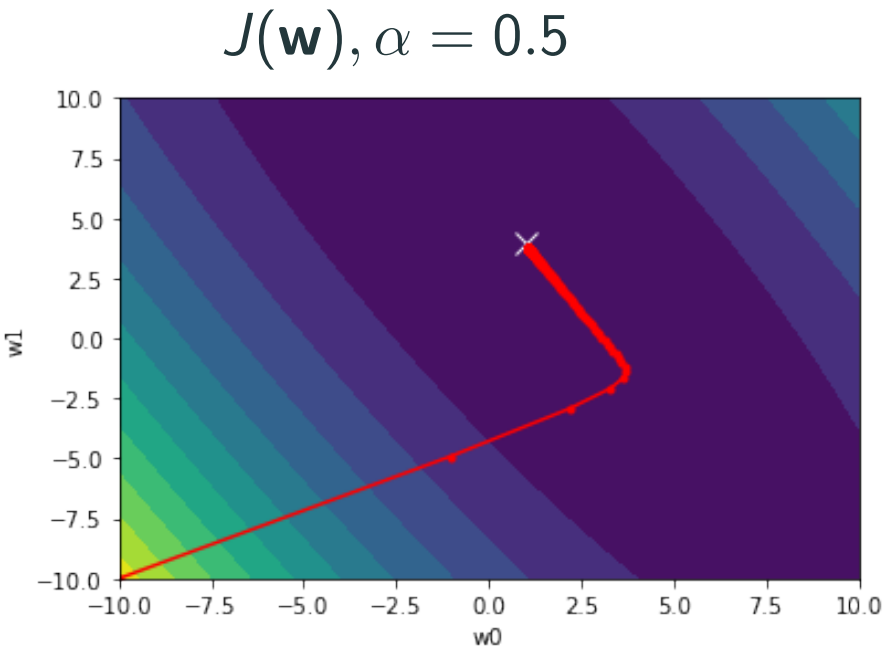
Back to our problem...

Applying 200 iterations of batch gradient descent to our toy problem, this is how our quadratic loss looks like

True function: $y = 5x + 1$

Estimated parameters:
$w_0 = 1.05, w_1 = 3.75$

close enough...



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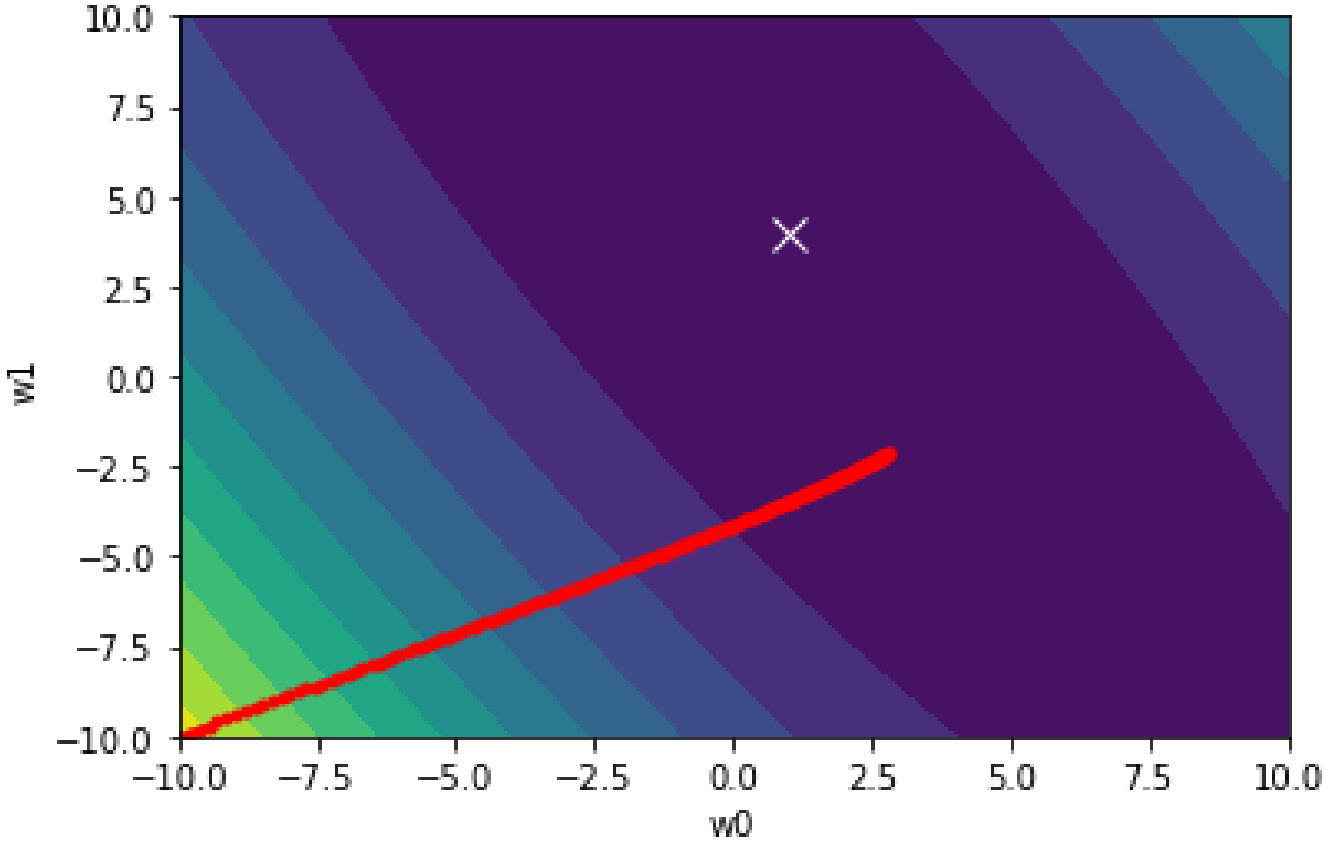
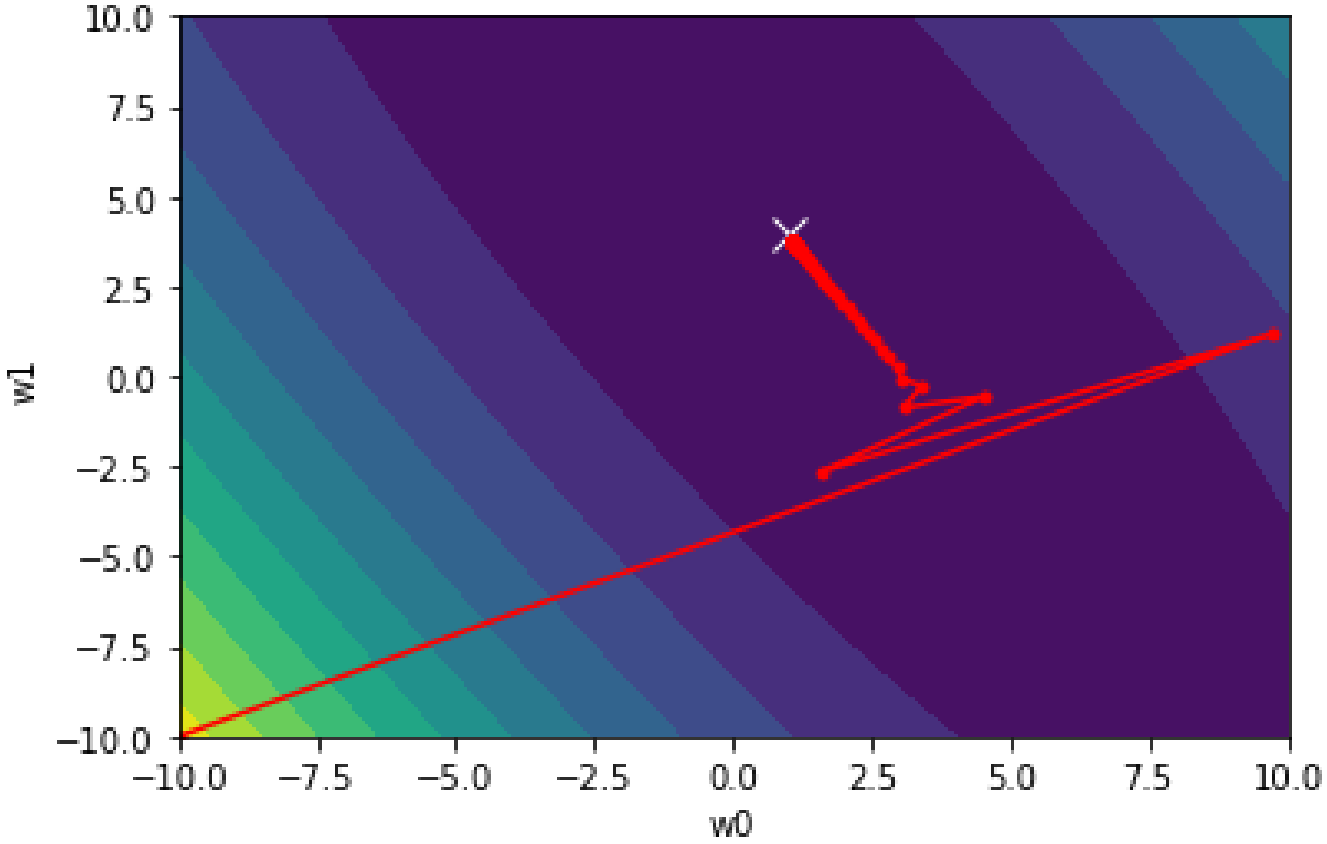
What happens if we choose a lower learning rate ($\alpha = 0.05$)?



Back to our problem...

What happens if we choose a higher learning rate($\alpha = 1.5$)?



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- yes!

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- where each column, $\mathbf{x}_j = [x_j^{(1)}, ..., x_j^{(n)}]$

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Our loss is:

$$\begin{aligned} J(\mathbf{w}) &= \frac{1}{2}\sum_{i=1}^{n}(y^{(i)} - \sum_{j}^{d} x_j^{(i)} w_j)^2 \\ &= \frac{1}{2}(\mathbf{y} - \mathbf{X}\mathbf{w})^T(\mathbf{y} - \mathbf{X}\mathbf{w}) \end{aligned}$$

Analytical solution

The gradient of the loss is

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and solving for $\mathbf{w}$, gives us

$$\mathbf{w} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$

Back to our problem again

Using the normal equation,

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Similar solution as gradient descent!

but rather easier ?

Recap

We've looked at two ways of optimising the parameters/weights in a linear regression setting:

- through gradient descent, leading to the batch gradient descent algorithm

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We've looked at two ways of optimising the parameters/weights in a linear regression setting:

- through gradient descent, leading to the batch gradient descent algorithm
- analytically, through the matrix normal equations
- both of these useful on their own, and key ingredients when solving more complex problems, as we'll see...

Conclusions

What have we learned today?

- Basics of supervised vs. unsupervised learning
- Linear models for regression
- The least means squares algorithm
- Analytical solution to the unconstrained linear regression problem

So... what next?

Tomorrow, we'll learn how to deal with more complex nonlinear problems, using the tools we've worked through today ;)